

# On the Stability of Localized Electrostatic Structures

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*Dedicated to Professor Dieter Pfirsch on his 60th Birthday*

The eigenvalue problem associated with transverse perturbations of localized one-dimensional electrostatic Vlasov equilibria is analyzed with respect to its spectral and variational properties.

**Key words:** Vlasov-Poisson equilibria, linear stability, unstable normal modes, nonlocality, Hermitian spectral operator.

## 1. Introduction

Experimentally one often encounters non-thermally excited plasmas in which turbulent fluctuations coexist with persistent coherent structures. The macroscopic properties of such a system, like the anomalous transport, are to a large extent unknown and a basic question is, for example, how far the plasma behaviour deviates from that of quasilinear treatment in which the coherent part is absent. A first step toward an understanding of such a highly developed plasma state is the stability analysis of the coherent nonlinear structure. If successful, such an analysis provides information about the normal mode spectrum of the system which replaces the Fourier mode spectrum in the description of weak turbulence in a homogeneous plasma. In the case of electrostatic coherent waves, however, a kinetic treatment is called for and the stability analysis is rendered intricate due to the nonlocal kinetic character of the linearized eigenvalue problem. Rigorous answers concerning stability are as a rule lacking as simplifications and approximations seem to be indispensable.

Although a number of ingredients of a stability theory has been presented in the past for special cases, such as the instability of neighbouring equilibria [1] or the formulation of stability criteria based on variational principles [2–4], a stability theory as a whole is far from being established. Besides the nonlocal property of the eigenvalue

problem, a further reason for the lack of a profound stability theory is that often distribution functions are in use which are in conflict with the existence requirements of the equilibrium. A typical example are distribution functions which are monotone decreasing functions of the isotropic energy  $\frac{1}{2} (m_j v^2) + e_j \phi$ .

The results derived in the present paper are exclusively based on realistic distribution functions.

In Sect. 2 a brief review is given on Vlasov equilibria that have been confirmed experimentally. In Sect. 3 the linearized eigenvalue problem is derived and the aperiodic nature of unstable solutions is shown in Sect. 4 generalizing an earlier proof [4]. Finally the spectral operator is analyzed in Sect. 5 making use of its representation as a differential operator of arbitrary order.

## 2. Electrostatic Vlasov Equilibria

Electrostatic structures of a two-component plasma are described by the normalized Vlasov-Poisson system

$$[\tau_s \partial_t + v \cdot \nabla + q_s \nabla \phi \cdot \partial_v] f_s = 0, \quad s = e, i, \quad (1)$$

$$\Delta \phi = \int d^2 v [f_e - f_i], \quad (2)$$

where the normalizations

$$\frac{e \phi}{k T_e} \rightarrow \phi, \quad \omega_{pe} t \rightarrow t,$$

$$v/v_{th} \rightarrow v, \quad x/\lambda_D \rightarrow x$$

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and definitions

$$q_s = \begin{cases} 1 \\ -\theta \end{cases}, \quad \tau_s = \begin{cases} 1 \\ \sqrt{\frac{m_i T_e}{m_e T_i}} \end{cases}, \quad (3)$$

$$\alpha_s = \begin{cases} 1, & s = e \\ \theta, & s = i \end{cases}$$

are adopted ( $\theta = T_e/T_i$ ).

Localized 1-D equilibrium solutions, satisfying

$$L_s f_{0s} = 0, \quad (4)$$

$$\phi_0''(x) = \int dv_x [f_{0e}(x, v_x) - f_{0i}(x, v_x)], \quad (5)$$

where

$$L_s = v_x \partial_x + q_s \phi_0'(x) \partial v_x \quad (6)$$

is the unperturbed Liouville operator, are known (see Ref. [5] and the references therein). They constitute the class of symmetric or asymmetric phase-space vortices (electron- and ion holes), several types of double layers or hydrodynamic-like solitons. For small amplitudes,  $\psi \ll 1$ , the electric equilibrium potentials can be written as

$$|\phi_0(x)| = \begin{cases} \psi \operatorname{sech}^4(\psi^{1/4} \kappa x) \\ \text{e-i-holes,} \end{cases} \quad (7a)$$

$$|\phi_0(x)| = \begin{cases} \frac{\psi}{4} [1 + \tanh(\psi^{1/2} \kappa x)]^2 \\ \text{SEADL, SIADL,} \end{cases} \quad (7b)$$

$$|\phi_0(x)| = \begin{cases} \psi \operatorname{sech}^2(\psi^{1/2} \kappa x) \\ \text{soliton,} \end{cases} \quad (7c)$$

where  $\kappa$  is some constant. The associated distributions are continuous functions in the phase space, a requirement that is often found to be violated in equilibrium theories. They are of the type

$$f_{0s}(x, v_x) = (2\pi)^{-1/2}$$

$$\begin{cases} \exp \left[ -\frac{1}{2} (\sigma \sqrt{2E_{||s}} + v_{0s})^2 \right] & E_{||s} > 0 \\ \exp \left[ -\alpha_s E_{||s} - \frac{1}{2} v_{0s}^2 \right] & E_{||s} \leq 0 \end{cases}, \quad (8)$$

and are functions of the constants of motion, namely the parallel energy  $E_{||s} = v_x^2/2 - q_s \phi_0(x)$  (where in the case of ions  $\phi_0$  is conveniently replaced by  $\phi_0 - \psi$ ) and the sign of the velocity of free

particles,  $\sigma = \operatorname{sgn} v_x$ . In the asymptotic region,  $\phi_0(x) \rightarrow 0$ , the distribution (8) becomes a shifted Maxwellian in the frame moving with the coherent wave and, hence, a Maxwellian in the laboratory frame. The drift velocities  $v_{0s}$  are amplitude dependent and are typically of order unity for small amplitude waves ( $\psi \ll 1$ ). They decrease with increasing amplitude. The number of particles, trapped in the electrostatic potential  $\phi_0(x)$  is represented by  $\alpha_s$ ;  $\alpha_s$  of the characteristic species turns out to be negative for holes and the associated double layers. Most of these structures have been confirmed recently in the laboratories, as summarized in [5]. A synopsis of localized electrostatic structures is given in the Table.

### 3. The Linearized Eigenvalue Problem

Linear transverse perturbations of these preferred equilibria are conveniently described by the ansatz

$$f_s = F_{0s}(x, v_x, v_y) + f_{1s}(x, v_x, v_y) e^{i(ky - \omega t)} + \text{c.c.}, \quad (9)$$

$$\phi = \phi_0(x) + \phi_1(x) e^{i(ky - \omega t)} + \text{c.c.}, \quad (10)$$

where  $F_{0s} = f_{0s} \frac{1}{\sqrt{2\pi}} e^{-E_{\perp}}$ ;  $E_{\perp} = v_y^2/2$  and  $k$  are the perpendicular energy and wavenumber, respectively.

It is worth noticing that  $F_{0s}$  depending now on  $v_x$  and  $v_y$  is neither a function of the isotropic energy,  $E = E_{||} + E_{\perp}$ , nor is it monotonically decreasing in  $E$ , two conditions suggested by the stability criterium derived from an energy principle.

By means of the transformation [1, 6]

$$g_s = f_{1s} + q_s \phi_1 \partial_{E_{||s}} F_{0s} \quad (11)$$

the following linearized set of equations is obtained for the perturbations:

$$[-i\hat{\omega}_s + L_s] g_s = q_s \frac{\hat{\omega}_s}{i} G_s \phi_1, \quad (12)$$

$$\Lambda \phi_1 = \int d^2v [g_e - g_i], \quad (13)$$

where  $\hat{\omega}_s$  is the Doppler shifted frequency,

$$\hat{\omega}_s = \tau_s \omega - k v_y. \quad (14)$$

$\Lambda$  is the field operator [6]

$$\Lambda = \partial_x^2 - k^2 + V''(\phi_0), \quad (15)$$

where  $V(\phi_0)$  is the pseudopotential representing the equilibrium and is related to  $\phi_0(x)$  through

$$\phi_0'(x)^2/2 + V(\phi_0) = 0. \quad (16)$$

$G_s$ , given by

$$G_s = \left[ \partial_{E_{\parallel s}} + \frac{kv_y}{\hat{\omega}_s} \partial_{E_{\perp}} \right] F_{0s}, \quad (17)$$

commutes with  $L_s$ :

$$G_s L_s = L_s G_s. \quad (18)$$

A solution of (12) is obtained by means of the method of characteristic and becomes [7]

$$g_s = -i\hat{\omega}_s q_s G_s e^{i\hat{\omega}_s \tau_E(x, \sigma)} \cdot \int_{x_0}^x dx' \frac{e^{-i\hat{\omega}_s \tau_E(x', \sigma)}}{v_x(x', E_{\parallel s}, \sigma)} \phi_1(x'), \quad (19a)$$

$$v_x(x, E_{\parallel s}, \sigma) := \sigma \sqrt{2[E_{\parallel s} + q_s \phi_0(x)]} \quad (20)$$

is the parallel particle velocity expressed by the constants of motion  $\sigma$  and  $E_{\parallel s}$ , and

$$\tau_E(x, \sigma) := \int_{x_0}^x \frac{dx'}{v_x(x', E_{\parallel s}, \sigma)} \quad (21)$$

is the transient time of the particle over the distance  $|x - x_0|$ , where  $x_0$  is some reference point to be specified for each orbit.

From (21) follows

$$v_x \partial_x \tau_E = 1. \quad (22)$$

Note that  $\phi_1$  enters in (19a) nonlocally, which is the origin of the above mentioned difficulties. This nonlocal character can be expressed in a different way by a differential operator of infinite order. Using (22) and

$$\begin{aligned} & e^{-i\hat{\omega}_s \tau_E(x', \sigma)} \\ &= \frac{i}{\hat{\omega}_s} v_x(x', E_{\parallel s}, \sigma) \partial_{x'} e^{-i\hat{\omega}_s \tau_E(x', \sigma)}, \end{aligned} \quad (23)$$

repeated partial integrations in (19a) yield

$$\begin{aligned} g_s &= q_s G_s \sum_{n=0}^{\infty} \left( \frac{v_x \partial_x}{i\hat{\omega}_s} \right)^n \phi_1(x) \\ &= q_s G_s \sum_{n=0}^{\infty} \left( \frac{L_s}{i\hat{\omega}_s} \right)^n \phi_1(x). \end{aligned} \quad (19b)$$

Representing a geometrical series, it can be expressed formally by

$$g_s = q_s G_s \left( 1 + \frac{iL_s}{\hat{\omega}_s} \right)^{-1} \phi_1(x), \quad (19c)$$

which is simply the inversion of (12).

Insertion of  $g_s$  into (13) results in the linearized eigenvalue problem

$$K(x, \omega, k) \phi_1(x) = 0, \quad (24)$$

where the spectral operator  $K$  can be written as

$$K(x, \omega, k) = \Lambda - W \quad (25)$$

with

$$W = \sum_s \alpha_s W_s,$$

and  $W_s$  may assume one of the forms

$$\begin{aligned} W_s &:= \int d^2v \frac{G_s}{(1 + iL_s/\hat{\omega}_s)} \\ &= \int d^2v G_s \sum_{n=0}^{\infty} \left( \frac{L_s}{i\hat{\omega}_s} \right)^n. \end{aligned} \quad (26)$$

Of course, also (19a) can be used for the definition of  $W_s$ . The second form of (26) is called the fluid limit [8], assuming in some sense the smallness of

$$\left| \frac{L_s}{i\hat{\omega}_s} \right| = \left| \frac{v_x \partial_x}{i\hat{\omega}_s} \right|.$$

#### 4. The Aperiodic Nature of Unstable Solutions

Extending a proof of Schindler et al. [4] to non-isotropic distribution functions, non-monotone in the energy, I am showing now that unstable modes are aperiodic, i.e.  $\omega = i\gamma$  if  $\gamma > 0$ .

For this reason I define the following scalar products, assuming square integrable functions:

$$(f, g) := \int dx d^2v f^* g, \quad (27a)$$

$$[u, v] := \int dx u^* v. \quad (27b)$$

Note that  $L_s$  is anti-Hermitian with respect to the scalar product (27a), i.e.  $(f, L_s g) = -(L_s f, g)$ , and that  $\Lambda$  is Hermitian with respect to (27b). Multiplication of (24) by  $\phi_1^*$  and integration over  $x$  yields

$$[\phi_1, \Lambda \phi_1] = \sum_s \alpha_s \left( \phi_1 \frac{G_s}{1 + iL_s/\hat{\omega}_s} \phi_1, \right) \quad (28)$$

$$= -4v \sum_s \frac{\alpha_s}{\tau_s^2} \cdot \left( \phi_1, \frac{(v_x D \partial_{E_{||s}} + k v_y \partial_{E_{\perp}}) F_{0s} (k v_y + v_x D)}{([+]^2 + \gamma^2)([-]^2 + \gamma^2)} \phi_1 \right), \quad (33)$$

Assuming  $\omega = v + i\gamma$ , where  $\gamma > 0$ , (28) can be split into real and imaginary parts:

$$[\phi_1, \Lambda \phi_1] = \sum_s \alpha_s \left( \phi_1, \frac{1}{\hat{D}^2 + \gamma^2} \left[ (\hat{v}_s \hat{D} + \gamma^2) \partial_{E_{||s}} + \frac{k v_y}{\tau_s} \hat{D} \partial_{E_{\perp}} \right] F_{0s} \phi_1 \right), \quad (29a)$$

$$0 = \sum_s \alpha_s \left( \phi_1, \frac{1}{\hat{D}^2 + \gamma^2} \left[ (\hat{D} - \hat{v}_s) \partial_{E_{||s}} - \frac{k v_y}{\tau_s} \partial_{E_{\perp}} \right] F_{0s} \phi_1 \right) \equiv H(v), \quad (29b)$$

where the following abbreviations have been used

$$\hat{D} = v - \frac{1}{\tau_s} (k v_y + v_x D), \quad (30a)$$

$$\hat{v}_s = v - k v_y / \tau_s, \quad (30b)$$

and  $D$  represents the Hermitian operator  $D = (1/i)\partial_x$ . Changing the sign of  $v_x$  and  $v_y$  in (29b) and remembering that the scalar product (27a) involves a 2- $D$  velocity integration which remains unchanged by the transformation  $v_{0s} \rightarrow -v_{0s}$ , one gets

$$H(v) = -H(-v), \quad (31)$$

from which follows

$$2H(v) = H(v) - H(-v). \quad (32)$$

Since  $\hat{D} - \hat{v}_s = -\frac{1}{\tau_s} v_x D$ , the r.h.s. of (32) becomes

$$H(v) - H(-v) = \sum_s \frac{\alpha_s}{\tau_s} \left( \phi_1, (v_x D \partial_{E_{||s}} + k v_y \partial_{E_{\perp}}) \cdot F_{0s} \left\{ \frac{1}{[+]^2 + \gamma^2} - \frac{1}{[-]^2 + \gamma^2} \right\} \phi_1 \right)$$

where  $[ \pm ] := v \pm \tau_s^{-1} (k v_y + v_x D)$ . Since in general the scalarproduct in (33) does not vanish, it follows from (29b), (32) and (33) that  $v = 0$ . Hence unstable modes within the class of square integrable functions are purely growing. If there is a transition from stability to instability by changing a continuous parameter, it goes through  $\omega = 0$ , i.e. the marginal mode is necessarily a neighbouring equilibrium [1, 4].

## 5. An Evaluation of the Spectral Operator $K$

The “fluid” approach of (26) is now applied to get a more explicit form of the spectral operator  $K$  making use of the smallness of  $\psi$  and of specific equilibrium properties.

From (26) I define

$$W_s = \sum_{n=0}^{\infty} W_{sn}, \quad (34)$$

where  $W_{sn}$  is given by

$$\begin{aligned} W_{sn} &= \int d^2v G_s \left( \frac{L_s}{i\hat{\omega}_s} \right)^n = i^{-n} \int d^2v \frac{L_s^n}{\hat{\omega}_s^n} G_s \\ &= i^{-n} \int d^2v \frac{[v_x \partial_x + q_s \phi'_0(x) \partial_{v_x}]^n}{(\tau_s \omega - k v_y)^n} G_s \\ &= D^n \int d^2v \frac{v_x^n [(\tau_s \omega - k v_y) \partial_{E_{||s}} + k v_y \partial_{E_{\perp}}] F_{0s}}{(\tau_s \omega - k v_y)^{n+1}}, \end{aligned}$$

where (17), (18) and  $D = (1/i)\partial_x$  have been used. Consequently  $W_{sn}$  can be written as

$$W_{s0} = \frac{-1}{q_s} n'_{0s}(\phi_0) + n_{0s} [1 + \zeta_s Z(\zeta_s)], \quad (36a)$$

$$\begin{aligned} W_{sn} &= D^n \left[ -\frac{1}{q_s} \Omega_{sn} \frac{d}{d\phi_0} \mathcal{M}_{sn} \right. \\ &\quad \left. + \mathcal{M}_{sn} \left( 1 + \frac{\zeta_s}{n} \frac{d}{d\zeta_s} \right) \Omega_{sn} \right], \quad n \geq 1 \end{aligned} \quad (36b)$$

where  $\Omega_{sn}$  and  $\mathcal{M}_{sn}$  are defined by

$$\mathcal{M}_{sn} = \int dv_x v_x^n f_{0s}, \quad (37)$$

$$\Omega_{sn} = \frac{1}{\sqrt{2\pi}} \int dv_y \frac{e^{-v_y^2/2}}{(\tau_s \omega - k v_y)^n}, \quad (38)$$

$$\text{and } \zeta_s := \frac{\tau_s \omega}{\sqrt{2} k}.$$

In (38) the Landau pole prescription is invoked assuming an adiabatic switching on of the perturbations. It is easily seen that the perpendicular velocity moment  $\Omega_{sn}$  is related to the plasma dispersion function  $Z(\zeta_s)$ :

$$\Omega_{sn} = \frac{1}{(n-1)!} \left( \frac{-1}{\sqrt{2} k} \right)^n Z^{(n-1)}(\zeta_s), \quad n \geq 1, \quad (39)$$

where  $Z^{(n)} := \frac{d^n}{d\zeta^n} Z(\zeta)$ . There thus holds

$$\Omega_{s0} = 1, \quad \Omega_{s1} = \frac{-1}{\sqrt{2} k} Z(\zeta_s),$$

$$\Omega_{s2} = \frac{1}{2k^2} Z'(\zeta_s) = \frac{-1}{k^2} [1 + \zeta_s Z(\zeta_s)]$$

etc.

The parallel velocity moment  $\mathcal{M}_{sn}$  can be evaluated as follows:

Multiplication of  $L_s f_{0s} = 0$  with  $v_x^{n-1}$  and integration yields

$$\frac{d}{dx} \mathcal{M}_{sn} = (n-1) q_s \phi'_0(x) \mathcal{M}_{sn-2} \quad (40)$$

and therefore

$$\frac{d}{d\phi_0} \mathcal{M}_{sn} = (n-1) q_s \mathcal{M}_{sn-2}. \quad (41)$$

For  $n = 0, 1$  one has

$$\mathcal{M}_{s0} = n_{0s}, \quad \mathcal{M}_{s1} = n_{\infty}^s v_{0s}, \quad (42)$$

where  $n_{0s}$  is the unperturbed  $x$ -dependent density of species  $s$ , and  $n_{\infty}^s v_{0s}$  is the corresponding particle flux, which is constant in  $x$ , i.e.  $d\mathcal{M}_{s1}/d\phi_0 = 0$  in accordance with (41). The quantity  $n_{\infty}^s$  is the normalized asymptotic density of untrapped particles in those region where trapped particles of the sort  $s$  are absent. It is unity for e- and i-holes and deviates from unity for  $DLs$ . Starting with

(42) one can successively build up by means of (41) the higher parallel moments. They are of the form

$$\mathcal{M}_{sn} = \varphi_{sn}(\phi_0) + \mathcal{M}_{sn}^0, \quad (43)$$

where  $\varphi_{sn}(\phi_0)$  is the  $x$ -dependent part and vanishes for  $\phi_0 \rightarrow 0$ .

The constant part can be written as

$$\begin{aligned} \mathcal{M}_{sn}^0 &:= \lim_{\phi_0 \rightarrow 0} \mathcal{M}_{sn} = \frac{n_{\infty}^s}{\sqrt{2\pi}} \int dv_x v_x^n e^{-1/2(v_x + v_{0s})^2} \\ &= n_{\infty}^s \sum_{v=0}^n \binom{n}{v} v_{0s}^v \frac{1}{\sqrt{2\pi}} \int dv_x v_x^{n-v} e^{-v_x^2/2}. \end{aligned}$$

Since the latter integral vanishes for  $n-v$  odd, one gets for  $n$  even ( $n = 2n'$ )

$$\mathcal{M}_{s2n'}^0 = \sum_{v'=0}^{n'} \binom{2n'}{2v'} v_{0s}^{2v'} (1; 2; n' - v') \quad (45)$$

and for  $n$  odd ( $n = 2n' + 1$ )

$$\mathcal{M}_{s2n'+1}^0 = \sum_{v'=0}^{n'} \binom{2n'+1}{2v'+1} v_{0s}^{2v'+1} (1; 2; n' - v'), \quad (46)$$

where  $(1; 2; m) = 1 \cdot 3 \cdot 5 \dots (2m-1)$  [Gröbner-Hofreiter, Integraltafel, Springer-Verlag Wien 1961].

The first parallel moments, dropping henceforth the constant  $n_{\infty}^s$ , are

$$\begin{aligned} \mathcal{M}_{s0}^0 &= 1, \\ \mathcal{M}_{s1}^0 &= -v_{0s}, \\ \mathcal{M}_{s2}^0 &= 1 + v_{0s}^2, \\ \mathcal{M}_{s3}^0 &= -v_{0s}(3 + v_{0s}^2), \\ \mathcal{M}_{s4}^0 &= 3 + 6v_{0s}^2 + v_{0s}^4, \dots \end{aligned} \quad (47)$$

They satisfy

$$\frac{d}{dv_{0s}} \mathcal{M}_{sn}^0(v_{0s}) = n \mathcal{M}_{sn-1}^0(v_{0s}), \quad (48)$$

which can be derived easily from the definition (44) of  $\mathcal{M}_{sn}^0$ . With these expressions the spectral operator  $K$  becomes

$$K = \partial_x^2 - k^2 + \sum_s \alpha_s \left\{ \frac{n_{0s}}{2} Z'(\zeta_s) - \sum_{n=1}^{\infty} D^n \left[ \Omega_{sn} \{ \mathcal{M}_{sn} - (n-1) \mathcal{M}_{sn-2} \} + \mathcal{M}_{sn} \frac{\zeta_s}{n} \Omega'_{sn}(\zeta_s) \right] \right\}, \quad (49)$$

which in the small amplitude case,  $\psi \ll 1$ , reduces to

$$K = \partial_x^2 - k^2 + \sum_s \alpha_s \left\{ \frac{1}{2} Z'(\zeta_s) - \sum_{n=1}^{\infty} D^n \left[ \Omega_{sn} \left( \mathcal{M}_{sn}^0 - \frac{d}{dv_{0s}} \mathcal{M}_{sn-1}^0 \right) + \mathcal{M}_{sn}^0 \frac{\zeta_s}{n} \Omega'_{sn}(\zeta_s) \right] \right\} \quad (50a)$$

$$= \partial_x^2 - k^2 + \sum_s \alpha_s \left\{ \frac{1}{2} Z'(\zeta_s) - D \frac{v_{0s}}{2\sqrt{2}k} Z''(\zeta_s) - \frac{D^2}{2k^2} \left[ v_{0s}^2 + (1 + v_{0s}^2) \frac{\zeta_s}{2} \frac{d}{d\zeta_s} \right] Z'(\zeta_s) + O(D^3) \right\}. \quad (50b)$$

Hence,  $K$  is given as a differential operator of arbitrary order. Equation (49) represents the expression for arbitrary equilibrium amplitudes, whereas (50a, b) applies for small amplitude equilibria. Several properties of  $K$  can now be derived from (49), (50):

1)  $K$  is generally non-Hermitian within the Landau prescription, the lack of self-adjointness being here due to the odd power terms of  $D = (1/i)\partial_x$ . These terms occur in connection with even derivatives of the plasma dispersion function which in the case of purely growing modes,  $\omega = i\gamma$ , carries an extra factor  $i$  noting that

$$Z(i\gamma_s) = i\sqrt{\pi} e^{\gamma_s^2} (1 - \operatorname{erf} \gamma_s)$$

with

$$\zeta_s = \frac{i\tau_s\gamma}{\sqrt{2}k} \equiv i\gamma_s.$$

This result conforms with an earlier statement of Lewis & Symon [6] saying that the general Hermiticity of their dispersion matrix  $D(\omega)$  is lost by the analytical continuation of  $D(\omega)$  involved in the Landau prescription, in contrast to the van Kampen form of treating the initial value problem which preserves the Hermiticity of  $D$ . We note in parenthesis that the dispersion matrix is a representation of the spectral operator  $K$  corresponding to a particular choice of orthonormal basis functions.

2)  $K$  is, however, self-adjoint for nonpropagating equilibria, implying vanishing particle fluxes. For, in this case,  $v_{0s} = 0$ , and the coefficients of  $D^{2n+1}$  vanish identically. This is in accord with an earlier result of Berk et al. [9] and others who explicitly used the orbit symmetry in the case of a standing wave pattern. Of course, self-adjointness is also implied for the restricted class of symmetric (or antisymmetric) square integrable functions. The Hermitian property of  $K$  then gives rise to a variational theorem for the normal mode frequency  $\omega_n$  and implies that if  $\eta_n(x)$  is an approximation of order  $\varepsilon$  to the normal potential  $\phi_1(x)$ , then the solution of  $[\eta_n, K\eta_n]$  for  $\omega_n$  will be accurate to order  $\varepsilon^2$  [6, 9].

3) The convergence property of (50) appears to be unaffected by the smallness parameter  $\varepsilon \sim \psi^{1/2}$  (or  $\psi^{1/4}$ , see (7)), which is introduced by  $\partial_x$ , assuming that the normal mode adopts the same scale length in  $x$  as the equilibrium. The former follows from the fact that  $D$  occurs always in the combination  $D/k$  and  $\varepsilon$ , connected with  $\partial_x$ , is always factored out by  $k$ , i.e. it vanishes by rescaling  $k$ . Hence the series does not possess an asymptotic character because the smallness parameter is lost.

4) It is an open question whether  $\eta_0(x)$ , the ground state of  $A$ , represents a reasonable approximation to  $\phi_1(x)$  in the fluid limit, especially if one wants to cut the series. A counter example is the case of a finite  $k$  neighbouring e-hole equilibrium where the calculation of the bilinear form  $[\eta_0, K\eta_0]$  gives rise to an alternating series with rapidly increasing coefficients. The expansion parameter  $|L_s/i\hat{\omega}_s|$  is of order unity and does not provide the desired convergence properties. In the case of a  $k=0$  neighbouring equilibrium, of course, the kinetic limit appears to be more appropriate.

In summary, investigating the eigenvalue problem associated with transverse electrostatic

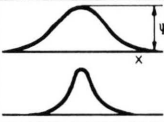
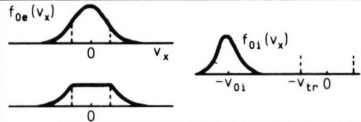
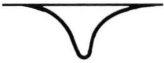
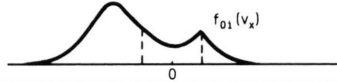

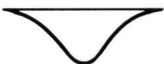
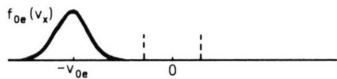

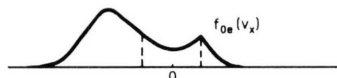


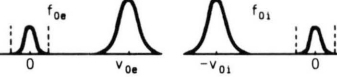

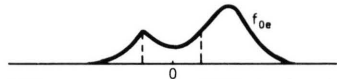

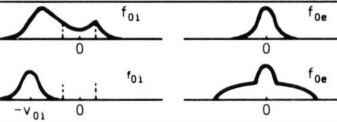
NAME	$\phi_0(x)$ (GRAPHICAL)	$\phi_0(x)$ (ANALYTICAL; $\Psi \ll 1$ )	RELEVANT DISTRIBUTIONS	TYPICAL PARAMETERS
ION ACOUSTIC SOLITON		$\phi_0(x) = \begin{cases} \Psi \operatorname{sech}^2 \sqrt{\frac{\Psi}{6}} x \\ \Psi \operatorname{sech}^4 \sqrt{\frac{b \sqrt{\Psi}}{15}} x \end{cases}$		ISOTHERMAL $\alpha_e = 1$ NON ISOTHERMAL $\alpha_e \leq 0$ $0 \leq \Psi \leq 1.26$ $0 \leq \Psi \leq 4.8$ ( $\alpha_e = 0$ ) $v_{01} \geq \sqrt{\theta}$ $\theta \gg 1$
ION HOLE (i-hole)		$\phi_0(x) = -\Psi \operatorname{sech}^4 \sqrt{\frac{b \sqrt{\Psi}}{15}} x$		$v_{01} \leq 1.3$ $\alpha_i < -0.71$ $\theta \gg 1$ $0 \leq \Psi \leq 1$
ASYMMETRIC ION HOLE				LIKE ION HOLE
GOULD- TRIVELPIECE SOLITON		$\phi_0(x) = -\Psi \operatorname{sech}^2 \sqrt{\frac{0.72\Psi}{2v_{0e}^2}} x$		$v_{0e} \geq 3 + \kappa_\perp^{-2}$ $\kappa_\perp = \frac{2.4}{R}$ CYL. WAVE TUBE
ELECTRON HOLE (e-hole)		$\phi_0(x) = \Psi \operatorname{sech}^4 \sqrt{\frac{b \sqrt{\Psi}}{15}} x$		$v_{0e} \leq 1.3$ $\alpha_e < -0.71$
ASYMMETRIC ELECTRON HOLE				LIKE ELECTRON HOLE
STRONG DOUBLE LAYER		NON EXISTENT FOR $\Psi \ll 1$		$v_{0e} > 1$ $v_{01} > \sqrt{\theta}$ $1 < \Psi (\leq 2000)$ $\alpha_e > 0, \alpha_i > 0$
SLOW ELECTRON ACOUSTIC DOUBLE LAYER (SEADL)		$\phi_0(x) = \frac{\Psi}{4} [1 + \tanh \kappa_e x]^2; \kappa_e \sim \sqrt{\Psi}$		$v_{0e} \leq 1.3$ $v_{01} \approx 0$ $\theta^{-1} > 3.5$ $\alpha_e < -0.71$ $\alpha_i > 0$
SIADL ION ACOUSTIC DOUBLE LAYER IADL		$\phi_0(x) = \begin{cases} \frac{-\Psi}{4} [1 - \tanh \kappa_i x]^2; \kappa_i \sim \sqrt{\Psi} \\ \frac{-\Psi}{2} [1 - \tanh \tilde{\kappa}_i x]; \tilde{\kappa}_i \sim \Psi \end{cases}$		$v_{0e} \approx 0$ $v_{01} \leq 1.3$ LIKE ION ACOUSTIC SOLITONS $\theta > 3.5$ $\alpha_e > 0$ $0 \leq \Psi \leq 1$ $\alpha_i < -0.71$ TWO-TEMPERATURE ELECTRONS OR ELECTRON BEAMS

Table 1. Members of the zoo of electrostatic one-dimensional structures together with some characteristic properties



perturbations of localized electrostatic Vlasov equilibria with one non-ignorable coordinate we gave a general proof of the aperiodic nature of un-

stable normal modes and derived several properties of the spectral operator including its general non-Hermiticity within a Landau prescription.

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